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# Connected Vertex Cover for $(sP_1 + P_5)$ -Free Graphs<sup>\*</sup>

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**Abstract.** The CONNECTED VERTEX COVER problem is to decide if a graph  $G$  has a vertex cover of size at most  $k$  that induces a connected subgraph of  $G$ . This is a well-studied problem, known to be NP-complete for restricted graph classes, and, in particular, for  $H$ -free graphs if  $H$  is not a linear forest. On the other hand, the problem is known to be polynomial-time solvable for  $sP_2$ -free graphs for any integer  $s \geq 1$ . We prove that it is also polynomial-time solvable for  $(sP_1 + P_5)$ -free graphs for every integer  $s \geq 0$ .

## 1 Introduction

A set  $S$  of vertices in a graph  $G$  forms a *vertex cover* of  $G$  if every edge of  $G$  is incident with a vertex of  $S$ . The set  $S$  is an *independent set* if no two vertices in  $S$  are adjacent. These definitions lead to two classical graph problems, which are both NP-complete: the VERTEX COVER problem is to decide if a given graph  $G$  has a vertex cover of size at most  $k$  for a given integer  $k$ ; the INDEPENDENT SET problem is to decide if a given graph  $G$  has an independent set of size at least  $\ell$  for a given integer  $\ell$ . A set  $S$  of at least  $k$  vertices of a graph  $G$  on  $n$  vertices is a vertex cover if and only if  $V_G \setminus S$  is an independent set (of size at most  $n - k$ ). Hence VERTEX COVER and INDEPENDENT SET are polynomially equivalent. A vertex cover of a graph  $G$  is connected if it induces a connected subgraph of  $G$ . In our paper, we focus on the corresponding decision problem.

CONNECTED VERTEX COVER

*Instance:* a graph  $G$  and an integer  $k$ .

*Question:* does  $G$  have a connected vertex cover  $S$  with  $|S| \leq k$ ?

In 1977, Garey and Johnson [9] proved that CONNECTED VERTEX COVER is NP-complete for planar graphs of maximum degree 4. More recently, Priyadarsini and Hemalatha [18] and Fernau and Manlove [8] strengthened this result to 2-connected planar graphs of maximum degree 4 and planar bipartite graphs of maximum degree 4, respectively. Wanatabe et al. [22] proved that CONNECTED VERTEX COVER is NP-complete even for 3-connected graphs. Very recently,

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Munaro [16] proved the same for line graphs of planar cubic bipartite graphs and for planar bipartite graphs of arbitrarily large girth, and Li et al. [13] showed NP-completeness for 4-regular graphs.

We now turn to tractable cases. Ueno et al. [21] proved that CONNECTED VERTEX COVER is polynomial-time solvable for graphs of maximum degree at most 3. Escoffier et al. [7] proved the same result for chordal graphs. As VERTEX COVER is also polynomial-time solvable for chordal graphs [10], the authors of [7] proposed a general study on the complexity of CONNECTED VERTEX COVER on graph classes for which VERTEX COVER is polynomial-time solvable. This leads us to the research question of our paper:

*For which classes of graphs do the complexities of VERTEX COVER and CONNECTED VERTEX COVER coincide?*

This question was addressed by Chiarelli et al. [6] who considered classes of graphs characterized by a single forbidden induced subgraph  $H$ . Such graphs are called  $H$ -free. They observed that the results of Munaro [16] imply that CONNECTED VERTEX COVER is NP-complete for  $H$ -free graphs if  $H$  contains a cycle or a claw. Using Poljak's construction [17], VERTEX COVER is readily seen to be NP-complete for graphs of arbitrarily large girth and thus for  $H$ -free graphs whenever  $H$  contains a cycle. When  $H$  is the claw, VERTEX COVER becomes polynomial-time solvable for  $H$ -free graphs [15,20]. Hence, there exist graphs  $H$  such that CONNECTED VERTEX COVER and VERTEX COVER have different complexities when restricted to  $H$ -free graphs (assuming  $P \neq NP$ ).

So the complexity of CONNECTED VERTEX COVER is known for  $H$ -free graphs unless  $H$  is a linear forest (the disjoint union of one or more paths). Even the case where  $H$  is a single path on  $r$  vertices (denoted  $P_r$ ) is settled neither for VERTEX COVER nor for CONNECTED VERTEX COVER; it is not known if there exists an integer  $r$  such that VERTEX COVER or CONNECTED VERTEX COVER is NP-complete for  $P_r$ -free graphs. Lokshtanov et al. [14] proved that INDEPENDENT SET, and thus VERTEX COVER, is polynomial-time solvable for  $P_5$ -free graphs. Recently, Grzesik et al. [11] extended this to  $P_6$ -free graphs. We also note that if VERTEX COVER is polynomial-time solvable on  $H$ -free graphs for some graph  $H$ , then it is polynomial-time solvable on  $(P_1 + H)$ -free graphs. This follows from the folklore observation that to solve the complementary problem of INDEPENDENT SET on a  $(P_1 + H)$ -free graph one solves the problem on each  $H$ -free graph obtained by removing a vertex and all its neighbours.

**Theorem 1 ([11]).** *For every  $s \geq 0$ , VERTEX COVER can be solved in polynomial time for  $(sP_1 + P_6)$ -free graphs.*

By using the concept of the price of connectivity [3,5,12], Chiarelli et al. [6] proved that CONNECTED VERTEX COVER is polynomial-time solvable for  $sP_2$ -free graphs for any integer  $s \geq 1$ . For VERTEX COVER this follows by combining two classical results [2,19] (as is well-known). No other complexity results are known for CONNECTED VERTEX COVER for  $H$ -free graphs if  $H$  is a linear forest.

**Our Contribution.** We continue the study of [6,7] and prove the following result, which includes polynomial-time solvability for  $P_5$ -free graphs.

**Theorem 2.** *For every  $s \geq 0$ , CONNECTED VERTEX COVER can be solved in polynomial time for  $(sP_1 + P_5)$ -free graphs.*

**Our Method.** It is easy to construct graphs with a minimum connected vertex cover that do not contain a minimum vertex cover; see the graph  $G_1$  in Fig. 1. We also note that the difference between a minimum vertex cover and a minimum connected vertex cover in an  $(sP_1 + P_5)$ -free graph is at most 3 if  $s = 0$  and at most  $3s + 10$  if  $s \geq 1$  [12]. We cannot exploit this property directly as that would require an algorithm to enumerate all minimum vertex covers in polynomial time. Moreover, the graph  $G_2$  in Fig. 1 shows that even if this were possible, it is not immediately obvious how to proceed; one cannot necessarily hope to find a minimum connected vertex cover by extending a minimum vertex cover. As an extra complication, for CONNECTED VERTEX COVER one cannot extend results on  $H$ -free graphs to results on  $(sP_1 + H)$ -free graphs in a straightforward way (certainly one cannot use the technique for VERTEX COVER described before Theorem 1).

Our method is based on an analysis of the structure of dominating sets in  $(sP_1 + P_5)$ -free graphs using a characterization of  $P_5$ -free graphs due to Bacsó and Tuza [1]. We translate the problem into a problem in which we try to extend a partial vertex cover into a full connected vertex cover. We solve this extension variant of CONNECTED VERTEX COVER by using Theorem 1 (applied to the smaller class of  $(sP_1 + P_5)$ -free graphs). We show how to do this in Section 3 and then show how to use this result to prove Theorem 2 in Section 4. An important ingredient of our proof is to reduce the size of the input graph by contracting an edge between two vertices  $u$  and  $v$  whenever we detect that  $u$  and  $v$  will belong to the connected vertex cover. This idea stems from the observation that a connected graph  $G$  on  $n$  vertices has a connected vertex cover of size  $k$  if and only if  $G$  contains the star  $K_{1,n-k}$  on  $n - k + 1$  vertices as a contraction.

## 2 Preliminaries

Let  $G = (V, E)$  be a graph. For a set  $S \subseteq V$ , the graph  $G[S]$  denotes the subgraph of  $G$  induced by  $S$ , and we say that  $S$  is *connected* if  $G[S]$  is connected. We write  $G - S = G[V \setminus S]$ , and if  $S = \{u\}$  we may simply write  $G - u$ . For a vertex  $u \in V$ , we write  $N_G(u) = \{v \mid uv \in E\}$  to denote the neighbourhood of  $u$ . For a set  $S \subseteq V$ , we write  $N_G(S) = (\bigcup_{u \in S} N_G(u)) \setminus S$ . A subset  $D \subseteq V$  is a *dominating* set of  $G$  if every vertex of  $V \setminus D$  is adjacent to at least one vertex of  $D$ . An edge  $uv$  of a graph  $G = (V, E)$  is *dominating* if  $\{u, v\}$  is dominating. The *contraction* of an edge  $uv \in E$  is the operation that replaces  $u$  and  $v$  by a new vertex adjacent to precisely those vertices of  $V \setminus \{u, v\}$  adjacent to  $u$  or  $v$  in  $G$ . Recall that for a graph  $H$ , we say that another graph  $G$  is  *$H$ -free* if it does not contain an induced subgraph isomorphic to  $H$ . The *disjoint union*  $G + H$  of two vertex-disjoint graphs  $G$  and  $H$  is the graph  $(V_G \cup V_H, E_G \cup E_H)$ . The disjoint union of  $r$  copies of a graph  $G$  is denoted by  $rG$ . A *linear forest* is the disjoint union of one or more paths. The following, straightforward lemma holds for any linear forest.

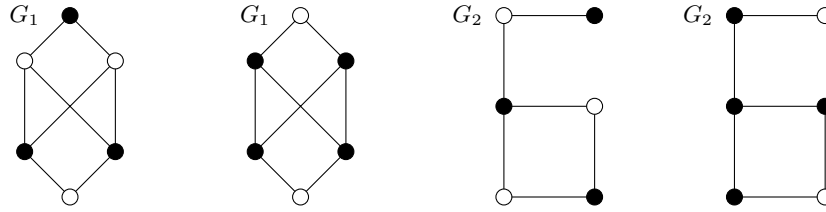


Fig. 1: An example of a  $P_5$ -free graph  $G_1$  with a minimum connected vertex cover (coloured black in the right-hand drawing) that contains no minimum vertex cover (there are exactly two, indicated by the sets of black and white vertices in the left-hand drawing). The graph  $G_2$  is an example of a  $(P_1 + P_5)$ -free graph with a minimum vertex cover (coloured black in the left hand drawing) that is not contained in any minimum connected vertex cover; clearly any connected vertex cover that contains it has at least five vertices and an example of a minimum connected vertex cover on four vertices is indicated by the vertices coloured black in the right-hand drawing.

**Lemma 1.** *Let  $G$  be a connected  $(sP_1 + P_5)$ -free graph for some  $s \geq 0$ . The graph obtained from  $G$  after contracting an edge is also connected and  $(sP_1 + P_5)$ -free.*

We will use the following result of Bacsó and Tuza [1] as a lemma.

**Lemma 2 ([1]).** *Every connected  $P_5$ -free graph  $G$  has a dominating set  $D$ , computable in  $O(n^3)$  time, that induces either a  $P_3$  or a complete graph.*

Note that it is not difficult to compute the set  $D$  in polynomial time; this also follows from a more general result of Camby and Schaudt [4] for  $P_r$ -free graphs ( $r \geq 1$ ).

Proofs of some lemmas are omitted due to space restrictions.

### 3 An Auxiliary Problem

In this section we prove that a variant of CONNECTED VERTEX COVER can be solved in polynomial time for  $(sP_1 + P_5)$ -free graphs for every integer  $s \geq 0$ .

To prove Theorem 2 we will solve a polynomial number of instances of this variant, which we show can be solved in polynomial time for  $(sP_1 + P_5)$ -free graphs for every  $s \geq 0$ . We introduce the variant by first describing its input. Let  $G$  be a connected graph, let  $J \subseteq V_G$  be a subset of the vertex set of  $G$  and let  $y$  be a vertex of  $J$ . We call the triple  $(G, J, y)$  *cover-complete* if it has the following properties (see also Fig. 2):

- (A)  $J$  is an independent set;
- (B)  $y$  is adjacent to every vertex of  $G - J$ ;
- (C) the neighbours of each vertex in  $J \setminus \{y\}$  form an independent set in  $G - J$ .

We now describe the problem.

## CONNECTED VERTEX COVER COMPLETION

*Instance:* a cover-complete triple  $(G, J, y)$ .

*Goal:* find a smallest connected vertex cover  $S$  of  $G$  such that  $J \subseteq S$ .

We will show how to solve this problem in polynomial time for  $(sP_1 + P_5)$ -free graphs for any  $s \geq 0$ .

Let  $(G, J, y)$  be a cover-complete triple, where  $G$  is a connected  $(sP_1 + P_5)$ -free graph. For a vertex  $w \in N_G(J \setminus \{y\})$ , we write  $J_w = N_G(w) \cap J$ . Note that, by (B),  $y \in J_w$ . Let  $G'$  be the graph obtained from  $G$  by contracting every edge of  $G[J_w \cup \{w\}]$ . As  $G[J_w \cup \{w\}]$  is connected, contracting its edges reduces it to a single vertex which we denote  $y_w$ . We say that we have *set-contracted*  $G$  into  $G'$  via  $w$  and that we *contracted*  $J_w \cup \{w\}$  into  $y_w$ .

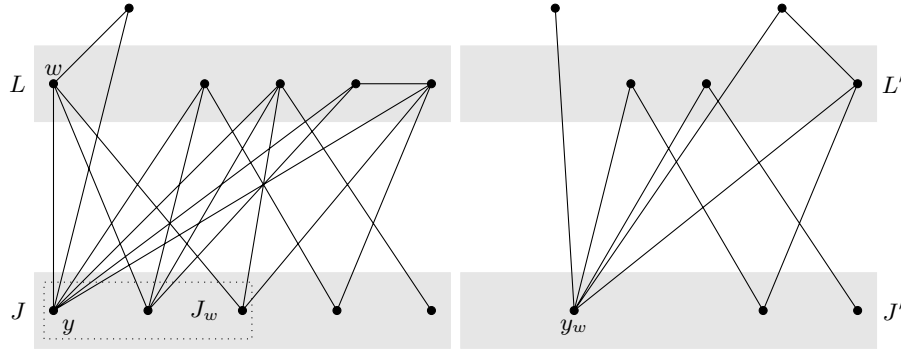


Fig. 2: An example of a cover-complete triple  $(G, J, y)$  and the cover-complete triple  $(G', J', y_w)$  obtained from set-contracting  $G$  via vertex  $w$ . The sets  $J' = (J \setminus J_w) \cup \{y_w\}$ ,  $L = N_G(J \setminus \{y\})$  and  $L' = N_{G'}(J' \setminus \{y_w\})$  are also displayed (the latter two sets will be formally introduced later).

**Lemma 3.** *Let  $(G, J, y)$  be a cover-complete triple, where  $G$  is a connected  $(sP_1 + P_5)$ -free graph for some  $s \geq 0$ . Let  $w \in N_G(J \setminus \{y\})$ , and let  $G'$  be the graph obtained from  $G$  after set-contracting via  $w$ . Let  $J' = (J \setminus J_w) \cup \{y_w\}$  and  $y' = y_w$ . Then the following hold:*

1.  $G'$  is a connected  $(sP_1 + P_5)$ -free graph;
2.  $(G', J', y')$  is a cover-complete triple;
3. A set  $S \subseteq V_G$  is a (smallest) connected vertex cover of  $G$  that contains  $J \cup \{w\}$  if and only if  $(S \setminus (J \cup \{w\})) \cup J'$  is a (smallest) connected vertex cover of  $G'$  that contains  $J'$ .

Let  $(G, J, y)$  be a cover-complete triple. We define  $L_J = N_G(J \setminus \{y\})$ . If there is no ambiguity, we will just write  $L = L_J$ . Note that, by (C),  $L$  is the union of a number of independent sets, but  $L$  itself might not be independent. However we can deduce the following lemma, which follows immediately from property (C).

**Lemma 4.** *Let  $(G, J, y)$  be a cover-complete triple. If  $w_1$  and  $w_2$  are two adjacent vertices in  $L$ , then no vertex of  $J \setminus \{y\}$  is adjacent to both  $w_1$  and  $w_2$ .*

We introduce two key definitions. Two vertices  $w_1, w_2 \in L$  form a *pseudo-dominating pair* if  $w_1$  and  $w_2$  are non-adjacent;  $w_1$  has a neighbour  $x_1 \in J$  not adjacent to  $w_2$ ; and  $w_2$  has a neighbour  $x_2 \in J$  not adjacent to  $w_1$ . Three vertices  $w_1, w_2, w_3 \in L$  form a *pseudo-dominating triple* if  $w_1$  is adjacent to neither  $w_2$  nor  $w_3$ ;  $w_2$  and  $w_3$  are adjacent;  $J$  contains two distinct vertices  $x_1$  and  $x_2$  such that  $x_1 \in N_G(w_1) \setminus N_G(\{w_2, w_3\})$  and  $x_2 \in (N_G(w_1) \cap N_G(w_2)) \setminus N_G(w_3)$ . See the illustrations in Fig. 3, from which we also observe that no pseudo-dominating pair or pseudo-dominating triple can be found in a  $P_5$ -free graph.

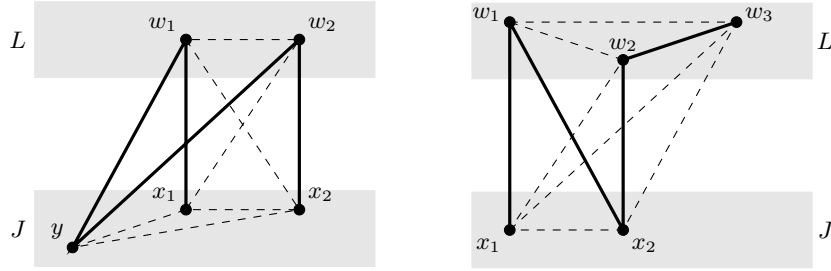


Fig. 3: Examples, on the left, of a pseudo-dominating pair  $(w_1, w_2)$ , and, on the right, of a pseudo-dominating triple  $(w_1, w_2, w_3)$ . As easily seen, the presence of either implies the existence of at least one induced  $P_5$ .

Let  $S$  be a connected vertex cover of  $G$  that contains  $J$ . Recall that  $J$  is an independent set. A subset  $L^* \subseteq L \cap S$  is a *connector* of  $S$  if  $J \cup L^*$  is connected.

**Lemma 5.** *Let  $(G, J, y)$  be a cover-complete triple, where  $G$  is an  $(sP_1 + P_5)$ -free graph for some  $s \geq 0$ . Let  $S$  be a connected vertex cover of  $G$  that contains  $J$ . If  $S$  contains both vertices of a pseudo-dominating pair  $w_1, w_2$ , then  $S$  has a connector of size at most  $s + 1$  that contains both  $w_1$  and  $w_2$ .*

**Lemma 6.** *Let  $(G, J, y)$  be a cover-complete triple, where  $G$  is an  $(sP_1 + P_5)$ -free graph for some  $s \geq 0$ . Let  $S$  be a connected vertex cover of  $G$  that contains  $J$ . If  $S$  contains all three vertices of a pseudo-dominating triple  $w_1, w_2, w_3$ , then  $S$  has a connector of size at most  $s + 2$  that contains  $\{w_1, w_2, w_3\}$ .*

Let  $(G, J, y)$  be a cover-complete triple. Let  $S$  be a connected vertex cover of  $G$  that contains  $J$ . If  $S$  contains both vertices of some pseudo-dominating pair of  $G$  or all three vertices of some pseudo-dominating triple of  $G$ , then  $S$  is of *type 1*. Otherwise  $S$  must contain at most one vertex of any pseudo-dominating pair and at most two vertices of any pseudo-dominating triple of  $G$ . In that case we say that  $S$  is of *type 2*. We observe that  $G$  might have connected vertex covers of only one type.

We will now see, in Lemma 8, how to find a smallest type 1 connected vertex cover of a graph  $G$  of a cover-complete triple  $(G, J, y)$  in polynomial time (if it exists). After that we shall prove how to find a smallest type 2 connected vertex cover of  $G$  in polynomial time (if it exists). To compute these sets we need the following lemma, which uses Theorem 1 in its proof.

**Lemma 7.** *Let  $(G, \{y\}, y)$  be a cover-complete triple, where  $G$  is an  $(sP_1 + P_5)$ -free graph for some  $s \geq 0$ . Then it is possible to compute a smallest connected vertex cover of  $G$  that contains  $y$  in polynomial time.*

Using Lemmas 5–7, we can now prove the following lemma.

**Lemma 8.** *Let  $(G, J, y)$  be a cover-complete triple. Then it is possible to find in polynomial time a smallest type 1 connected vertex cover of  $G$ .*

Let  $(G, J, y)$  be a cover-complete triple. Using Lemma 8 we can find a smallest type 1 connected vertex cover of  $G$ . However, it might be possible that  $G$  has a smaller connected vertex cover of type 2. To investigate this, we introduce two reduction rules that will transform a cover-complete triple  $(G, J, y)$  into a triple  $(G', J', y')$  with  $|J'| < |J|$ . We say that such a rule is *safe* if the following holds:

1. If  $G$  is  $(sP_1 + P_5)$ -free and connected, then  $G'$  is  $(sP_1 + P_5)$ -free and connected.
2.  $(G', J', y')$  is cover-complete.
3. Given a smallest connected vertex cover  $S'$  of  $G'$  that contains  $J'$ , it is possible, in polynomial time, to find a smallest connected vertex cover  $S$  of  $G$  that contains  $J$ .

**Rule 1.** Set-contract via  $x$  whenever  $x$  is a vertex in  $L \cap N_G(w_1) \cap N_G(w_2)$  for some pseudo-dominating pair  $(w_1, w_2)$ .

**Rule 2.** For any vertex  $w_5 \in L$  that is not adjacent to any vertex of a clique of four vertices  $w_1, w_2, w_3, w_4$  in  $L$ , delete  $w_5$  and set-contract via  $u$  for every  $u \in L \cap N_G(w_5)$ .

**Lemma 9.** *Rules 1 and 2 are safe.*

We call a cover-complete triple  $(G, J, y)$  *free* if  $G$  has no pseudo-dominating pair with a common neighbour in  $L$ , and moreover,  $G[L]$  is  $(P_1 + K_4)$ -free. By exhaustively applying Rules 1 and 2 in arbitrary order, which we may safely do due to Lemma 9, we have the following lemma.

**Lemma 10.** *A cover-complete triple  $(G, J, y)$  can be modified, in polynomial time, into a free cover-complete triple  $(G', J', y)$  with the following properties:*

1. *If  $G$  is  $(sP_1 + P_5)$ -free and connected, then  $G'$  is  $(sP_1 + P_5)$ -free and connected.*
2. *Given a smallest connected vertex cover  $S'$  of  $G'$  that contains  $J'$ , it is possible to find in polynomial time a smallest connected vertex cover  $S$  of  $G$  that contains  $J$ .*



Let  $(G, J, y)$  be a free cover-complete triple. A connector of a connected vertex cover  $S$  of  $G$  is *minimal* if it does not properly contain a smaller connector of  $S$ .

**Lemma 11.** *Let  $(G, J, y)$  be a free cover-complete triple that has a pseudo-dominating pair  $(w_1, w_2)$ . Then every minimal connector  $L^*$  of every type 2 connected vertex cover  $S$  of  $G$  has size at most 5.*

**Lemma 12.** *Let  $(G, J, y)$  be a free cover-complete triple that has no pseudo-dominating pair. It is possible to find in polynomial time a clique  $K \subseteq L$  with  $N_G(K) \cap J = J$ .*

We are now ready to prove the following theorem.

**Theorem 3.** *For every  $s \geq 0$ , CONNECTED VERTEX COVER COMPLETION can be solved in polynomial time for  $(sP_1 + P_5)$ -free graphs.*

*Proof.* Let  $s \geq 0$  and let  $(G, J, y)$  be a cover-complete triple, where  $G$  is an  $(sP_1 + P_5)$ -free graph. We first apply Lemma 10 to obtain a free cover-complete triple  $(G', J', y')$  in polynomial time. By the same lemma,  $G'$  is  $(sP_1 + P_5)$ -free. Our aim is to find a smallest connected vertex cover of  $G'$  that contains  $J'$  in polynomial time, so that we can apply statement 2 of Lemma 10. We first compute in polynomial time a smallest type 1 connected vertex cover  $S^*$  of  $G'$  using Lemma 8. We now need to compute a smallest type 2 connected vertex cover  $S'$  of  $G'$  and compare  $|S'|$  with  $|S^*|$ .

First suppose that  $G'$  contains a pseudo-dominating pair. We guess a minimal connector of size at most 5 and apply Lemma 3 on its vertices. (By guess, we mean choose a set of up to 5 vertices and test to see if they form a minimal connector. We eventually look at all such sets.) If we obtain an instance of the form  $(G'', \{y''\}, y'')$ , then we apply Lemma 7. Then we uncontract all contracted edges to get a connected vertex cover of  $G'$  of type 2. By Lemma 11, doing this for every guessed minimal connector of size at most 5 gives us a smallest type 2 connected vertex cover  $S'$  of  $G'$ . As we process each guess in polynomial time and there are at most  $O(n^5)$  guesses, we find  $S'$  in polynomial time. We compare  $S'$  and  $S^*$  and choose the smaller of the two.

Now suppose that  $G'$  has no pseudo-dominating pair. Let  $L' = N_{G'}(J' \setminus \{y'\})$ . By Lemma 12, we can obtain in polynomial time a clique  $K \subseteq L'$  with  $N_{G'}(K) \cap J' = J'$ . Let  $K = \{w_1, \dots, w_r\}$  for some  $r \geq 1$ . As  $K$  is a clique, every vertex cover contains at least  $r - 1$  vertices of  $K$ . We will do as follows: first we will find in polynomial time a smallest connected vertex cover of  $G'$  that contains  $J' \cup K$ , and then we will find in polynomial time, for  $i = 1, \dots, r$ , a smallest connected vertex cover of  $G'$  that contains  $J' \cup (K \setminus \{w_i\})$  and that does not contain  $w_i$ . As there are  $O(n)$  cases, the total time is polynomial.

We start by computing a smallest connected vertex cover of  $G'$  that contains  $J' \cup K$  by set-contracting via each vertex of  $K$ . By Lemma 3, this yields a cover-complete triple  $(G'', \{y''\}, y'')$  to which we apply Lemma 7. Then we uncontract all contracted edges in polynomial time. By Lemma 3, this yields a smallest connected vertex cover  $S_K$  of  $G'$  that contains  $J' \cup K$ .

We now show how to compute, in polynomial time, a smallest connected vertex cover of  $G'$  that contains  $J' \cup (K \setminus \{w_1\})$  and that does not contain  $w_1$ . The case  $i \geq 2$  is done in the same way.

Let  $A = L' \setminus N_{G'}(w_1)$  consist of all non-neighbours of  $w_1$  in  $L'$ . As  $G'[L']$  is  $(K_4 + P_1)$ -free by definition, we find that  $G'[A]$  is  $K_4$ -free. As  $w_1$  is not in the connected vertex cover we are looking for we remove  $w_1$ , and we set-contract via each neighbour of  $w_1$  in  $L$ . By Lemma 3, we may now consider the resulting cover-complete triple  $(G'', J'', y'')$  where  $G''$  is connected and  $(sP_1 + P_5)$ -free. As  $G'$  had no pseudo-dominating pairs, we have that  $G''$  has no pseudo-dominating pairs. We write  $L'' = N_{G''}(J'' \setminus \{y''\})$ . As  $L'' \subseteq A$ , we find that  $G''[L'']$  is  $K_4$ -free.

*Claim. Every minimal connector  $L^*$  of every connected vertex cover of  $G''$  that contains  $J''$  has size at most 3.*

We prove the claim by showing that  $L^*$  is a clique, which implies that  $L^*$  has size at most 3, as  $G''[L'']$  is  $K_4$ -free. Suppose instead that  $L^*$  is not a clique. Then  $L^*$  contains two non-adjacent vertices  $w_1$  and  $w_2$ . As  $L^*$  is a minimal connector,  $w_1$  has a neighbour in  $J''$  not adjacent to  $w_2$ , and vice versa. But then  $(w_1, w_2)$  is a pseudo-dominating pair of  $G''$ : this is not possible, as  $G''$  has no pseudo-dominating pairs. This contradiction proves the claim.

We now guess a minimal connector by considering all subsets in  $L''$  that have size at most 3. For each guess we apply Lemma 3 on its vertices. If we obtain an instance  $(G''', \{y'''\}, y''')$ , then we apply Lemma 7. Then we uncontract all contracted edges to obtain in polynomial time a connected vertex cover of  $G''$  that contains  $J''$ . We take the smallest one of these connected vertex covers of  $G''$ . For this connected vertex cover of  $G''$ , we uncontract all contracted edges again to obtain in polynomial time a smallest connected vertex cover  $S_{w_1}$  of  $G'$  that contains  $J' \cup (K \setminus \{w_1\})$  and that does not contain  $w_1$ .

As mentioned, we pick the smallest one out of the connected vertex covers  $S_K$  and  $S_{w_i}$ ,  $1 \leq i \leq r$ , to obtain a smallest type 2 connected vertex cover of  $G'$ , the size of which we compare with the size of  $S^*$ . We pick the smallest one.

Thus we obtain in polynomial time a smallest connected vertex cover of  $G'$  that contains  $J'$  (both in the case where  $G'$  has a pseudo-dominating pair and in the case where  $G'$  has no pseudo-dominating pair). As stated, it remains to apply statement 2 of Lemma 10 to find in polynomial time a smallest connected vertex cover of  $G$  that contains  $J$ . The correctness of our algorithm follows immediately from the above case analysis and the description of the cases.  $\square$

## 4 Our Main Result

In this section we prove Theorem 2. We need two more lemmas (we use Lemma 2 to prove the first one).

**Lemma 13.** *Let  $s \geq 0$  and let  $G$  be a connected  $(sP_1 + P_5)$ -free graph. Then  $G$  has a connected dominating set  $D$  that is either a clique or has size at most  $2s^2 + s + 3$ . Moreover,  $D$  can be found in  $O(n^{2s^2+s+3})$  time.*

**Lemma 14.** *Let  $J$  be an independent set in a connected graph  $G$  such that  $J$  has a vertex  $y$  that is adjacent to every vertex of  $G - J$ . Let  $J'$  consist of those vertices of  $J \setminus \{y\}$  that have two adjacent neighbours in  $G - J$  (or equivalently, in  $G$ ). Then a subset  $S$  is a connected vertex cover of  $G$  that contains  $J$  if and only if  $S \setminus J'$  is a connected vertex cover of  $G - J'$  that contains  $J \setminus J'$ .*

We are now ready to prove our main result.

**Theorem 2. (Restated)** *For every  $s \geq 0$ , CONNECTED VERTEX COVER can be solved in polynomial time for  $(sP_1 + P_5)$ -free graphs.*

*Proof.* Let  $G$  be an  $(sP_1 + P_5)$ -free graph for some  $s \geq 0$ . We may assume without loss of generality that  $G$  is connected. By Lemma 13 we can first compute in  $O(n^{2s^2+s+3})$  time a connected dominating set  $D$  that either has size at most  $2s^2 + s + 3$  or is a clique. We note that, if  $D$  is a clique, any vertex cover of  $G$  contains all but at most one vertex of  $D$ . This leads to a case analysis where we guess the subset  $D^* \subseteq D$  of vertices not in a minimum connected vertex cover of  $G$ . Because  $|D^*| \leq 2s^2 + s + 3$ , the number of guesses is polynomial. For each guess of  $D^*$ , we compute a smallest connected vertex cover  $S_{D^*}$  that contains all vertices of  $D \setminus D^*$  and no vertex of  $D^*$ . Then, in the end, we return one that has minimum size overall.

Let  $D^*$  be a guess. We first show the following claim (proof omitted).

*Claim 1. We may assume without loss of generality that  $D \setminus D^*$  is connected.*

**Case 1.**  $D^* = \emptyset$ .

We compute a minimum vertex cover  $S'$  of  $G - D$  in polynomial time by Theorem 1. Clearly  $S' \cup D$  is a vertex cover of  $G$ . As  $D$  is a connected dominating set,  $S' \cup D$  is a connected vertex cover of  $G$ . Let  $S_\emptyset = S' \cup D$ . As  $S'$  is a minimum vertex cover of  $G - D$ ,  $S_\emptyset$  is a smallest connected vertex cover of  $G$  that contains all vertices of  $D$ . We remember  $S_\emptyset$ , which we found in polynomial time.

**Case 2.**  $1 \leq |D^*| \leq |D|$  (recall that  $|D| \leq 2s^2 + s + 3$ ).

Recall that we are looking for a smallest connected vertex cover of  $G$  that contains every vertex of  $D \setminus D^*$  but does not contain any vertex of  $D^*$ . Hence  $D^*$  must be an independent set and  $G - D^*$  must be connected (if one of these conditions is false, then we stop considering the guess  $D^*$ ). Moreover, a vertex cover that contains no vertex of  $D^*$  must contain all vertices of  $N_G(D^*)$ . Hence we can safely contract not only any edge between two vertices of  $D \setminus D^*$ , but also any edge between two vertices in  $N_G(D^*)$  or between a vertex of  $D \setminus D^*$  and a vertex in  $N_G(D^*)$ . We perform edge contractions recursively and as long as possible while remembering all the edges that we contract. Let  $G^*$  be the resulting graph.

Note that the set  $D^*$  still exists in  $G^*$ , as we did not contract any edges with an endpoint in  $D^*$ . By Claim 1, the set  $D \setminus D^*$  in  $G$  corresponds to exactly one vertex of  $G^*$ . We denote this vertex by  $y$ . We observe the following equivalence.

*Claim 2. Every smallest connected vertex cover of  $G^*$  that contains  $y$  and that does not contain any vertex of  $D^*$  corresponds to a smallest connected vertex*

cover of  $G$  that contains  $D \setminus D^*$  and that does not contain any vertex of  $D^*$ , and vice versa.

As we obtained  $G^*$  in polynomial time, and we can uncontract all contracted edges in polynomial time as well, Claim 2 tells us that we may consider  $G^*$  instead of  $G$ . As  $G$  is connected and  $(sP_1 + P_5)$ -free,  $G^*$  is connected and  $(sP_1 + P_5)$ -free as well by Lemma 1.

We write  $J^* = N_{G^*}(D^*)$  and note that  $y$  belongs to  $J^*$  as  $D$  is connected in  $G$ . We now consider the graph  $G^* - D^*$ . As  $G - D^*$  is connected,  $G^* - D^*$  is connected. By Claim 2, our new goal is to find a smallest connected vertex cover of  $G^* - D^*$  that contains  $J^*$ . By our procedure,  $J^*$  is an independent set of  $G^* - D^*$ . As  $D$  dominates  $G$ , we find that  $D \setminus D^*$  dominates every vertex of  $G - D^*$  that is not adjacent to a vertex of  $D^*$ . Hence the vertex  $y$ , which corresponds to the set  $D \setminus D^*$ , is adjacent to every vertex of  $(G^* - D^*) - J^*$  in the graph  $G^* - D^*$ .

Let  $J \subseteq J^*$  consist of  $y$  and those vertices in  $J^*$  whose neighbourhood in  $G^* - D^*$  is an independent set. As  $y$  is adjacent to every vertex of  $(G^* - D^*) - J^*$  in  $G^* - D^*$ , and we can remember the set  $J^* \setminus J$ , we can apply Lemma 14 and remove  $J^* \setminus J$ . That is, it suffices to find a smallest connected vertex cover of the graph  $G' = (G^* - D^*) - (J^* \setminus J)$  that contains  $J$ .

As  $J^*$  is an independent set of  $G^* - D^*$ , we find that  $J$  is an independent set of  $G'$ . By definition,  $y \in J$ . As  $y$  is adjacent to every vertex of  $(G^* - D^*) - J^*$  in  $G^* - D^*$ , we find that  $y$  is adjacent to every vertex in  $G' - J$ . By definition, the neighbours of each vertex in  $J \setminus \{y\}$  form an independent set in  $G' - J$ . Hence the triple  $(G', J, y)$  is cover-complete. This means that we can apply Theorem 3 to find in polynomial time a smallest connected vertex cover  $S'$  of  $G'$  that contains  $J$ .

We translate  $S'$  in polynomial time into a smallest connected vertex cover  $S^*$  of  $G^* - D^*$  that contains  $J^*$  by adding  $J^* \setminus J$  to  $S'$ . We translate  $S^*$  in polynomial time into a smallest connected vertex cover  $S_{D^*}$  of  $G$  that contains no vertex of  $D^*$  by uncontracting any contracted edges.

As mentioned, in the end we pick, in polynomial time, a smallest set of the sets  $S_{D^*}$ . This set is then a minimum connected vertex cover of  $G$ , which is obtained in polynomial time. We have not sought to optimize the running time of the algorithm so do not provide a detailed analysis, but observe that, for sufficiently large  $s$ , it is  $n^{O(s^3)}$ . The running time is dominated by obtaining a connected  $D \setminus D^*$  (in Claim 1). As  $D \setminus D^*$  has  $O(n^{2s^2+s+3})$  components and the paths required to join them each have  $O(s)$  vertices, the time required to find them is  $n^{O(s^3)}$ . The correctness of our algorithm follows immediately from the above case analysis and the description of the cases.  $\square$

## 5 Future Work

We pose two open problems. First, determine the complexity of CONNECTED VERTEX COVER for  $P_6$ -free graphs. Second, is there an integer  $r$  such that CONNECTED VERTEX COVER is NP-complete for  $P_r$ -free graphs?

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